SIMILARITY

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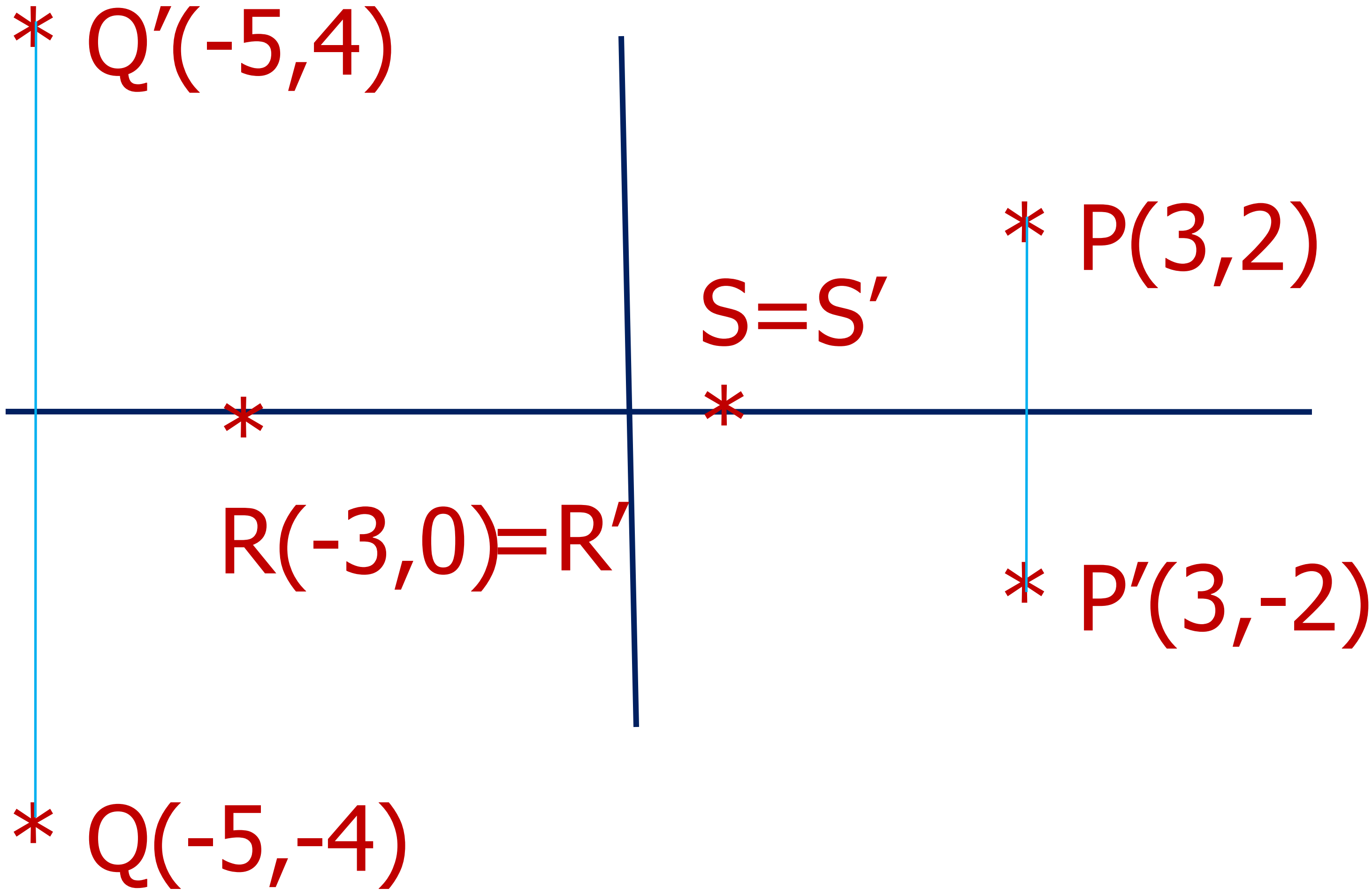
Plan

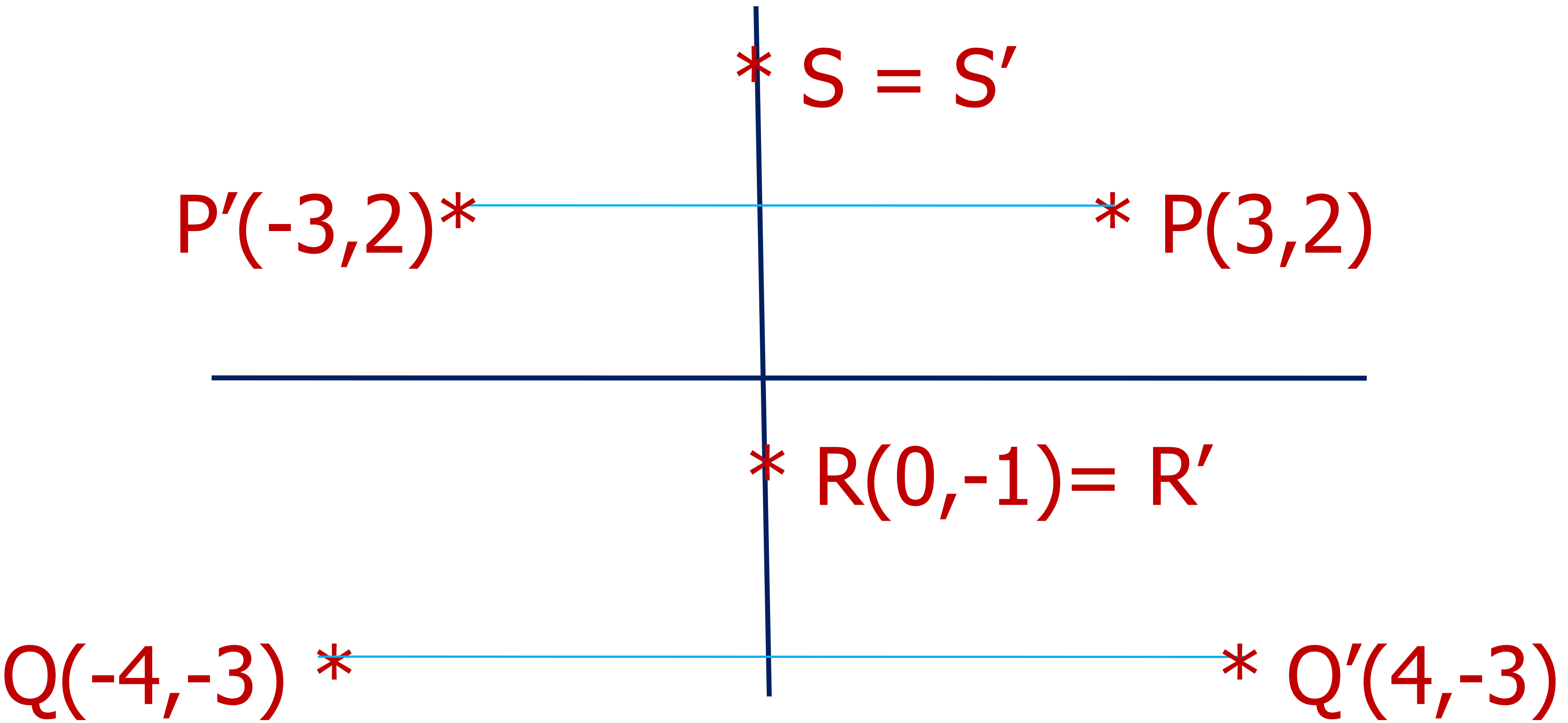
- Invariance point/line
- Eigen value and eigen vectors
- Similarity
- Orthogonal similarity
- Minimum polynomial

Reflection Transformation to X-axis



R unchanged, R called invariant point.
All of the points on X-axis are <u>invariant point</u>

Reflection Transformation to Y-axis



R unchanged, R called invariant point.
All of the points on Y-axis are <u>invariant point</u>

Observe that on a transformation, it is possible there are any points or lines that unchanged by the transformation. Points or lines that unchanged by the transformations called invariant points or invariant lines.

Eigen value and eigen vector

Suppose ***X***, ***Y*** $\in R^n$ and $\boldsymbol{A}_{nxn}$ is transformation matrix.

Generally behave ***Y=AX***, it is mean that by transformation matrix ***A***, a vector ***X*** tranformed to a vector ***Y***.

Suppose ***A*** = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, a vector ***X*** = $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

So ***AX*** = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ = ***Y***

In special case, it is possible that a vector ***X*** transformed to the vector ***X*** (or its multiple), so : ***AX*** = λ***X***, where λ scalar on field F. In this case, ***X*** called eigen vector or invariant vector (vektor tetap) or characteristics vector.

Supppose ***A*** = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, if a vector ***X*** = $\begin{pmatrix} 0 \\ -7 \end{pmatrix}$

Then ***AX*** = $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -7 \end{pmatrix} = \begin{pmatrix} 0 \\ -7 \end{pmatrix}$ = ***X***

How to get a invariant vector ***X***, such that:

$$\boldsymbol{AX} = \lambda \boldsymbol{X}$$

- $\boldsymbol{AX} = \lambda \boldsymbol{X}$ (1)

  $\boldsymbol{AX} - \lambda \boldsymbol{X} = 0$

  $\boldsymbol{AX} - \lambda \boldsymbol{IX} = 0$

  $(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{X} = 0$ (2)
- System (2) will be had a nontrivial solution (penyelesaian tidak nol) ***X***, if :

  $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$ (3)
- Expanding equation (3) will be got a polynomial in λ, that is $P(\lambda)$, it is called **characteristics polynomial**; thus characteristics polynomial is $P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}|$

- Because ***A*** and ***I*** are given, from the equation (3) where P(λ) = 0, it will be got values λ that called **eigen values** (akar karakteristik).
- Furthermore, if the value λ subtitute to the equation (2), it will be got **eigen vectors** (vektor invarian / vektor karakteristik) ***X***.

example

- Suppose a transformation matrix ***B*** = $\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$

  find eigen values and eigen vectors of the transformation matrix ***B***!

Solution :

$$\boldsymbol{B} - \lambda \boldsymbol{I} = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix}$$

$$|\boldsymbol{B} - \lambda \boldsymbol{I}| = \begin{vmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{vmatrix} = \{(-1 - \lambda)(4 - \lambda)\} - (2)(3)$$

$$|\boldsymbol{B} - \lambda \boldsymbol{I}| = (-2 - \lambda)(5 - \lambda)$$

$$P(\lambda) = |\boldsymbol{B} - \lambda \boldsymbol{I}| = 0$$

$$(-2 - \lambda)(5 - \lambda) = 0$$

$\lambda_1 = -2$ and $\lambda_2 = 5$ →Eigen value/

→akar karakteristik

Finding a eigen vectors can be done as follow:

For λ = -2, then :

$$(\boldsymbol{B} - \lambda \boldsymbol{I})\,\boldsymbol{X} = \begin{pmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \overset{H}{\sim} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

n= 2
r = 1; number of free var=1

A New equation:
$x_1 + 2x_2 = 0$

Free variable: x_2
suppose $x_2 = \alpha \rightarrow x_1 = -2\alpha$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

So, the first eigen vector $\mathbf{X}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

For $\lambda = 5$, then :

$$(\boldsymbol{B} - \lambda \boldsymbol{I})\ \boldsymbol{X} = \begin{pmatrix} -1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix} \overset{H}{\sim} \begin{pmatrix} -6 & 2 \\ 0 & 0 \end{pmatrix}$$

n= 2

r = 1; number of free var=1

A new equation: $-6x_1 + 2x_2 = 0$,

suppose $x_2 = \alpha \rightarrow x_1 = \frac{1}{3}\alpha$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3}\alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$$

Thus, the second eigen vector $\mathbf{X}_2 = \begin{pmatrix} \frac{1}{3} \\ 1 \end{pmatrix}$

practice

- Suppose a transformation matrix ***B*** = $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

  find eigen values and eigen vectors of the transformation matrix ***B***!

Problems

- Find eigen value and eigen vector of transformation matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$$

Find eigen value and eigen vector of transformation matrix

$$\boldsymbol{B} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 1 \\ -2 & 1 & -1 \end{pmatrix}$$

Find eigen value and eigen vector of transformation matrix

$$F = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Properties :

1) If λ_1, λ_2,…, and λ_k are the different eigen values and associated to the invariant vectors $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, …, $\boldsymbol{X}_k$ respectively, then the invariant vectors are linearly independent.

2) Eigen values of the transformation matrix $\boldsymbol{A}$ and $\boldsymbol{A}^T$ are equal.

example

- Suppose a transformation matrix ***A*** = $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$

  find eigen values and eigen vectors of the transformation matrix ***A***!

Solution:

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix}$$

$$|\boldsymbol{A} - \lambda \boldsymbol{I}| = \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = (5-\lambda)(1-\lambda)(1-\lambda)$$

$$P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$$

$$P(\lambda) = (5-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda_{1,2} = 1 \quad \text{and} \quad \lambda_3 = 5$$

Next step, finding invariant vector,

For λ = 1, then:

$$(\boldsymbol{A} - \lambda \boldsymbol{I})\,\boldsymbol{X} = \begin{pmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{pmatrix} \boldsymbol{X} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \overset{H}{\sim} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

n= 3
r = 1; number of free var=2

A new equation :
$x_1 + 2x_2 + x_3 = 0$ → Free variables are: x_2 and x_3

A new equation : $x_1 + 2x_2 + x_3 = 0$

suppose:

$x_2 = \alpha$ and $x_3 = \beta$ → then $x_1 = -2\alpha - \beta$

$$\text{Thus, } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} -2\alpha - \beta \\ \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

There are **two independent invariant vectors** :

$$\mathbf{X}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{X}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For λ = 5, then :

$$(\boldsymbol{A}-\lambda \boldsymbol{I})\,\boldsymbol{X}=\begin{pmatrix}2-\lambda & 2 & 1\\ 1 & 3-\lambda & 1\\ 1 & 2 & 2-\lambda\end{pmatrix}\boldsymbol{X}=\begin{pmatrix}-3 & 2 & 1\\ 1 & -2 & 1\\ 1 & 2 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}=0$$

$$\begin{pmatrix}-3 & 2 & 1\\ 1 & -2 & 1\\ 1 & 2 & -3\end{pmatrix}\overset{H}{\sim}\begin{pmatrix}1 & -2 & 1\\ -3 & 2 & 1\\ 1 & 2 & -3\end{pmatrix}\overset{H}{\sim}\begin{pmatrix}1 & -2 & 1\\ 0 & -4 & 4\\ 0 & 4 & -4\end{pmatrix}\overset{H}{\sim}\begin{pmatrix}1 & -2 & 1\\ 0 & -4 & 4\\ 0 & 0 & 0\end{pmatrix}$$

n= 3 ; r = 2;

number of free var = 1 →

A New equations:

$$x_1 - 2x_2 + x_3 = 0$$

$$-4x_2 + 4x_3 = 0$$

A new equations : $x_1 - 2x_2 + x_3 = 0$

$$-4x_2 + 4x_3 = 0$$

Free variable is x_3;

suppose: $x_3 = \alpha$ → then $x_2 = \alpha$, and $x_1 = \alpha$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} \alpha \\ \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Thus, the third invariant vector $\boldsymbol{X}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Problems

Find eigen value and eigen vector of transformation matrix

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

Find eigen value and eigen vector of transformation matrix

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix}$$

Find eigen value and eigen vector of transformation matrix

$$\boldsymbol{A} = \begin{pmatrix} -2 & 1 & 0 \\ -8 & 4 & 0 \\ -12 & 4 & 1 \end{pmatrix}$$

Find eigen value and eigen vector of transformation matrix

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find eigen value and eigen vector of transformation matrix

$$\boldsymbol{C} = \begin{pmatrix} -3 & -9 & -12 \\ 1 & 3 & 4 \\ 0 & 0 & 1 \end{pmatrix}$$

similarity

Recall your mind

- Find eigen value and eigen vector of transformation matrix

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- example:

$$\boldsymbol{A} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix} \quad \text{and } \boldsymbol{B} = \begin{pmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{are similar}$$

becuase there is non singular matrix

$$\boldsymbol{P} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

, such that $\boldsymbol{B} = \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}$.

Similarity

- Two Transformations matrix ***A*** and ***B*** called similar if there is nonsingular matrix ***R*** such that

$$\boldsymbol{B} = \boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R}$$

Properties

1) Two similar matrices have same eigen values.

2) if ***Y*** is eigen vector of ***B*** which is associated to eigen value λ_i, then ***X*** = ***RY*** is eigen vector of ***A*** which is associated to eigen value λ_i.

The current problems :

If there are any two matrix **A** and **B**, how to get a nonsingular matrix **P** such that

$$\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{B} \ ?$$

- Now, please find eigen values and eigen vectors of

$$\boldsymbol{D} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

Diagonal Matrix

$D = \begin{pmatrix} -3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{pmatrix}$. Eigen value of matrix D are $\lambda_1 = -3$, $\lambda_2 = 7$, and $\lambda_3 = 9$. Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Every *nxn* diagonal matrix have *n* eigen vectors which are linearly independent.

Similar to Diagonal Matrices: Diagonalization of Matrices

- Theorem:

  every *n-dimensional* matrix A which have *n* eigen vectors that linearly independence are similar to diagonal matrix.

Proof:

Suppose $X_1, X_2, X_3, \ldots, X_n$ are invariant vectors which are associated to the eigen value $\lambda_1, \lambda_2, \lambda_3, .., \lambda_n$ such that

$AX_i = \lambda_i X_i \ (i = 1, 2, 3, \ldots, n)$.

Suppose $P = [X_1 \ X_2 \ X_3 \ldots \ X_n]$, then

$AP = A[X_1 \ X_2 \ X_3 \ \ldots \ X_n] = [AX_1 \ AX_2 \ AX_3 \ \ldots \ AX_n]$

$AP = [\lambda_1 X_1 \ \lambda_2 X_2 \ \lambda_3 X_3 \ \ldots \ \lambda_n X_n]$

proof…(*cont*..)

$$AP = [X_1 \;\; X_2 \;\; X_3 \;\; \ldots \;\; X_n] \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 0 & \lambda_3 & .. & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

$$AP = [X_1 \;\; X_2 \;\; X_3 \;\; \ldots \;\; X_n]\, \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, .., \lambda_n)$$

$$AP = P\,D$$

$$P^{-1}AP = P^{-1}P\,D$$

$$P^{-1}AP = D$$

Thus, ***A*** similar to diagonal matrix D, because there is nonsingular matrix P such that $P^{-1}AP = D$.

- A matrices $\boldsymbol{A}_{nxn}$ which have *n* invariant vectors that linearly independence called *diagonalizable* (it can be diagonalized / similar to diagonal matrices).

Algorithm for diagonalization of matrix *Anxn* :

(1) find eigen value of matrix ***A***, that are λ_i (i =1,2, 3, ..., n)

(2) Find eigen vectors of matrix ***A*** which are associated to eigen value λ_i.

(3) if a number of eigen vectors < n, then matrix ***A*** is not diagonalizable. Finish.

(4) If a number of eigen vectors = n, then matrix ***A*** is **diagonalizable**; next step:

(4.1). Take $\boldsymbol{P} = [\boldsymbol{X}1\ \ \boldsymbol{X}2\ \ \boldsymbol{X}3\ \ \ldots\ \ \boldsymbol{X}\mathrm{n}]$, where $\boldsymbol{X}_i$ are eigen vectors of matrix ***A***.

(4.2). Find $\boldsymbol{P}^{-1}$

(4.3). $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, .., \lambda_n)$, with λ_i are eigen value of matrix ***A***.

(4.4). Finish.

example

Is a matrix $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$ diagonalizable ?. if yes, find matrices P such that $P^{-1}AP = D$ (diagonal).

Solution:

- Finding eigen value: $|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$

  thus the eigien value are $\lambda_1 = 1$ and $\lambda_2 = 4$.

- Finding eigen vectors:

  For λ = 1, then $(A - \lambda I)X = 0$

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

  the first eigen vector is $\boldsymbol{X}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

For λ = 4, then (**A** – λI)X = 0

$$\begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

the second eigen vector is $\boldsymbol{X}_2 = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$

Because matrix ***A*** have <u>two independent invariant vectors</u>, the matrix **A** is diagonalizable.

- **P** = ($\boldsymbol{X}_1$ $\boldsymbol{X}_2$) = $\begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$

- Finding inverse of P

$$P^{-1} = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

- It can be verified that : **P^{-1} A P = D**.

$$P^{-1} A P = \begin{pmatrix} -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} -1 & \frac{1}{2} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} = D.$$

example

- Is a matrix **B** = $\begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ diagonalizable ?

  if yes, find a nonsingular matrices **P** such that $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \mathbf{D}$ (Diagonal).

Solution:

From slide 21 to slide 26, it can be observed:

- Eigen value are $\lambda_{1,2} = 1$ and $\lambda_3 = 5$.
- For $\lambda = 1$, the eigen vectors are :

$$\boldsymbol{X}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{X}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For $\lambda = 5$, the eigen vector is :

$$\boldsymbol{X}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Because a matrix **B** have three independent invariant vectors, the matrix **B** is **diagonalizable**.

- **P** = ($\boldsymbol{X}_1$ $\boldsymbol{X}_2$ $\boldsymbol{X}_3$) = $\begin{pmatrix} -2 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
- $\mathbf{P}^{-1} = \dfrac{1}{4}\begin{pmatrix} -1 & 2 & -1 \\ -1 & -2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$
- $\mathbf{P}^{-1}\mathbf{B}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix} = \mathbf{D}$.

example

- Is a matrix **C** = $\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ 2 & 1 & 2 \end{pmatrix}$ diagonalizable?.

  If yes, find a matrix **P** such that $\mathbf{P}^{-1}\mathbf{C}\mathbf{P} = \mathbf{D}$ (Diagonal).

Solution:

- The eigen value of **C** are $\lambda_1 = 1$ and $\lambda_{2,3} = 2$.
- For $\lambda = 1$, the eigen vector is $\boldsymbol{X}_1 = \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}$

  For $\lambda = 2$, the eigen vector is $\boldsymbol{X}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

- Because the matrix **C** are only have two independent vectors, **the matrix C is not diagonalizable.**

practice

- Is a matrix A = $\begin{pmatrix} 3 & -2 \\ -2 & 6 \end{pmatrix}$ diagonalizable ?

  if yes, find a nonsingular matrices P such that $P^{-1}AP = D$ (Diagonal).

practice

Is a matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$ diagonalizable ?. if yes, find matrices P such that $P^{-1}AP = D$ (diagonal).

practice

- Is ***F*** diagonalizable?, if yes, find matrices ***R*** such that $\boldsymbol{R}^{-1}\boldsymbol{F}\boldsymbol{R} = \boldsymbol{D}$

$$F = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

practice

- Is a matrix $B = \begin{pmatrix} 2 & -4 & 2 \\ -4 & 2 & -2 \\ 2 & -2 & -1 \end{pmatrix}$ diagonalizable ?

  if diagonalizable, find a nonsingular matrices P such that $P^{-1}BP = D$ (Diagonal).

problem

- Is a matrix ***C*** diagonalizable ?. If diagonaliozable, find a matrices ***R*** such that ***R***$^{-1}$***CR*** = ***D.***

$$C = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

Orthogonal Similarity

Similarity of Symmetric matrices

- If $\boldsymbol{A}$ is a symmetric matrices, that is $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, it can be found an orthogonal matrices $\boldsymbol{R}$ such that $\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} = \boldsymbol{D}$ (diagonal).
- Because if $\boldsymbol{R}$ is orthogonal, $\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}}$ thus $\boldsymbol{R}^{-1}\boldsymbol{A}\boldsymbol{R} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{R} = \boldsymbol{D}$

Theorem

Eigen vectors of a symmetric matrices which come from the different eigen value are orthogonal

Proof:

Suppose $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are invariant vectors which are associated to λ_1 and λ_2 (where $\lambda_1 \neq \lambda_2$) of symmetric matrices A, then :

$$AX_1 = \lambda_1 X_1$$

$$X_2^T AX_1 = X_2^T \lambda_1 X_1 \qquad (1)$$

$$(X_2^T AX_1)^T = (\lambda_1 X_2^T X_1)^T$$

$$X_1^T A^T X_2 = \lambda_1 X_1^T X_2$$

$$X_1^T A X_2 = \lambda_1 X_1^T X_2 \qquad (2)$$

proof…(*cont*..)

Meanwhile :

$AX_2 = \lambda_2 X_2$

$X_1^T AX_2 = \lambda_2 X_1^T X_2$ (3)

$(X_1^T AX_2)^T = (\lambda_2 X_1^T X_2)^T$

$X_2^T A^T X_1 = \lambda_2 X_2^T X_1$

$X_2^T A X_1 = \lambda_2 X_2^T X_1$ (4)

from (2) and (3) :

$\lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$

$\lambda_1 X_1^T X_2 - \lambda_2 X_1^T X_2 = 0$

$(\lambda_1 - \lambda_2) X_1^T X_2 = 0$ or $X_1^T X_2 = 0$

This mean that X_1 orthogonal to X_2.

Consequence…

- For a symmetric matrices, The theorem above assure of an existence of orthogonal matrices $\boldsymbol{R}$, such that $\boldsymbol{R}^{T}\boldsymbol{A}\boldsymbol{R} = \boldsymbol{D}$.

For a symmetric matrices *A*, algorithm to find orthogonal matrices R such that $R^{-1}AR = D$ (diagonal) as follow:

(1) Find eigen value of matrix ***A***

(2) Find invariant vectors of matrix ***A***

(3) If all of the eigen values are different, then invariant vectors X1, .., Xn are orthogonal.

(3.1) normalized X1, ..., Xn become new vectors Y1, Y2,..., Yn.

(3.2) orthogonal matrix ***R*** = [Y1 Y2 ... Yn].

(3.3) $\boldsymbol{R}^{-1}\boldsymbol{AR} = \boldsymbol{D}$ (diagonal). finish.

(4) **If there are same eigen value**, suppose $\lambda_1 = \lambda_2$, then eigen vector X1 and X2 are not orthogonal; but eigen vectors X3, ..., Xn are orthogonal.

(4.1) apply Gram-Schmidt process to X1 and X2 to get a new orthogonal vectors W1 and W2.

(4.2) take W3 = X3, ..., Wn = Xn

(4.3) normalized vectors W1, W2, W3, ..., Wn become a new vectors Y1, Y2, Y3, .., Yn.

(4.4) orthogonal matrix ***R*** = [Y1 Y2 ... Yn].

(4.5) $\boldsymbol{R}^{-1}\boldsymbol{AR} = \boldsymbol{D}$ (diagonal). finish.

example

- find an orthogonal matrices ***R*** such that ***R***$^{-1}$***AR*** = ***D*** (diagonal), if

$$A = \begin{pmatrix} -4 & -6 \\ -6 & 1 \end{pmatrix}$$

Solution:

- Observe that **A** is symmetric, because $\mathbf{A}^T = \mathbf{A}$
- The eigen value of **A** are $\lambda_1 = 5$ and $\lambda_2 = -8$
- For $\lambda_1 = 5$, the eigen vector $\mathbf{X}_1 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$

  For $\lambda_2 = -8$, the eigen vector $\mathbf{X}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix}$
- Observe that $\mathbf{X}_1$ and $\mathbf{X}_2$ are orthogonal, $\mathbf{X}_1 . \mathbf{X}_2 = 0$.
- Normalize $\mathbf{X}_1$ and $\mathbf{X}_2$

$$\mathbf{g}_1 = \frac{X_1}{\|X_1\|} = \begin{pmatrix} -\frac{2}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} \end{pmatrix}, \text{ and } \mathbf{g}_2 = \frac{X_2}{\|X_2\|} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \end{pmatrix}$$

- Orthogonal matrix **R** = ($\mathbf{g}_1$ $\mathbf{g}_2$) = $\begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$
- $\mathbf{R}^{-1} = \mathbf{R}^T = \begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}$

- It can be verified that $\mathbf{R}^{-1}\mathbf{A}\mathbf{R} = \mathbf{D}$

$$\begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix}\begin{pmatrix} -4 & -6 \\ -6 & 1 \end{pmatrix}\begin{pmatrix} -\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -8 \end{pmatrix}$$

example

- find orthogonal matrices $\boldsymbol{P}$ such that $\boldsymbol{P}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{P} = \boldsymbol{D}$ (diagonal), if

$$\boldsymbol{L} = \begin{pmatrix} 7 & -2 & 1 \\ -2 & 10 & -2 \\ 1 & -2 & 7 \end{pmatrix}$$

Solution:

- Observe that **L** is symmetric, because $\mathbf{L}^T = \mathbf{L}$
- The eigen value of **L** are $\lambda_{1,2} = 6$ and $\lambda_3 = 12$
- For $\lambda = 6$, the eigen vector $\mathbf{X}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{X}_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

  For $\lambda = 12$, the third eigen vector $\mathbf{X}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

- Observe that $\mathbf{X}_1$ and $\mathbf{X}_3$ are orthogonal, $\mathbf{X}_1.\mathbf{X}_3 = 0$; beside $\mathbf{X}_2$ and $\mathbf{X}_3$ are also orthogonal, $\mathbf{X}_2.\mathbf{X}_3 = 0$. Meanwhile $\mathbf{X}_1$ and $\mathbf{X}_2$ are not orthogonal, why?

- Now, use gram-schmidt process to make $\mathbf{X}_1$ and $\mathbf{X}_2$ orthogonal:

$$w_1 = \mathbf{X}_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and } w_2 = \mathbf{X}_2 - \frac{X_2 . X_1}{X_1 . X_1} X_1 = \begin{pmatrix} -\frac{1}{5} \\ \frac{2}{5} \\ 1 \end{pmatrix}$$

and also take $w_2 = \mathbf{X}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ (why?).

- Normalize w_1, w_2, and w_3 become g_1, g_2, and g_3 respectively, thus:

$$g_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \\ 0 \end{pmatrix}, \quad g_2 = \begin{pmatrix} -\frac{1}{\sqrt{30}} \\ \frac{2}{\sqrt{30}} \\ \frac{5}{\sqrt{30}} \end{pmatrix}, \text{ and } g_3 = \begin{pmatrix} \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}$$

- $\mathbf{P} = (g_1\ g_2\ g_3) = \begin{pmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{30}} & -\frac{2}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{pmatrix}$

- It can be shown that $\mathbf{P^T L P} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 12 \end{pmatrix} = \mathbf{D}$

practice

- find an orthogonal matrices ***R*** such that ***$R^{-1}AR$ = D (diagonal)***

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

problem

- find an orthogonal matrices ***R*** such that ***R***^{T}***BR* = *D (diagonal)***

B= $$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

practice

- find an orthogonal matrices ***R*** such that $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{R} = \boldsymbol{D}$ ***(diagonal)***

$$\mathbf{C} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

problems

- find an orthogonal matrices ***R*** such that $\boldsymbol{R}^{-1}\boldsymbol{E}\boldsymbol{R} = \boldsymbol{D}$ **(diagonal)**

$$\boldsymbol{E} = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 4 \end{pmatrix}$$

- find an orthogonal matrices $\boldsymbol{R}$ such that $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{R} = \boldsymbol{D}$ (diagonal), if

$$\boldsymbol{F} = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 4 & -1 \\ 1 & -1 & 4 \end{pmatrix}$$

- find an orthogonal matrices ***R*** such that $\boldsymbol{R}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{R} = \boldsymbol{D}$ (diagonal), if

$$\boldsymbol{G} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

- find orthogonal matrices $\boldsymbol{R}$ such that $\boldsymbol{R}^{-1}\boldsymbol{H}\boldsymbol{R} = \boldsymbol{D}$ (diagonal), if

$$\boldsymbol{H} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$$

- find orthogonal matrices $\boldsymbol{R}$ such that $\boldsymbol{R}^{-1}\boldsymbol{K}\boldsymbol{R} = \boldsymbol{D}$ (diagonal), if

$$\boldsymbol{K} = \begin{pmatrix} 7 & -4 & -4 \\ -4 & 1 & -8 \\ -4 & -8 & 1 \end{pmatrix}$$

Minimum Polynomial

Characteristics Polynomial

Suppose given a nxn matrix $\boldsymbol{A}$:

- Characteristics Polynomial of $\boldsymbol{A}$ is

  $P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}|$

- From properties of adjoint matrices:

  $$\boldsymbol{A}\ \mathrm{adj}(\boldsymbol{A}) = |\boldsymbol{A}|\ \boldsymbol{I}$$

- Analogy : $(\boldsymbol{A} - \lambda \boldsymbol{I})\ \mathrm{adj}(\boldsymbol{A} - \lambda \boldsymbol{I}) = |\boldsymbol{A} - \lambda \boldsymbol{I}|\ \boldsymbol{I}$

  $$(\boldsymbol{A} - \lambda \boldsymbol{I})\ \mathrm{adj}(\boldsymbol{A} - \lambda \boldsymbol{I}) = P(\lambda)\ \boldsymbol{I}$$

Theorem:

- Every *n*-dimensional square matrices ***A*** is zero value of it characteristic polynomial.

→Thus, if $P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}|$, then $P(\boldsymbol{A}) = \boldsymbol{0}$.

example:

- $\boldsymbol{A} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}$
- $P(\lambda) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = (2-\lambda)(1-\lambda) - 6 = \lambda^2 - 3\lambda - 4$
- $P(\boldsymbol{A}) = \boldsymbol{A}^2 - 3\boldsymbol{A} - 4\boldsymbol{I} = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}^2 - 3\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} - 4\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- $P(\boldsymbol{A}) = \begin{pmatrix} 10 & 9 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} 6 & 9 \\ 6 & 3 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \boldsymbol{0}.$

Minimum Polynomial

- Suppose P(λ) is characteristics polynomial of square matrices ***A***, if m(λ) which is least degree of those polynomial such that m(***A***) = ***0***, then m(λ) called minimum polynomial of ***A***.

Algorithm for Finding Minimum Polynomial

- If $\mathbf{A} = a_o I$, then $m(\lambda) = \lambda - a_o$
- If $\mathbf{A} \neq a_o I$ for all a_o, but $\mathbf{A}^2 = a_1 \mathbf{A} + a_o I$, then $m(\lambda) = \lambda^2 - a_1 \lambda - a_o$.
- If $\mathbf{A}^2 \neq a_1 \mathbf{A} + a_o I$ for all a_1 and a_0, but $\mathbf{A}^3 = a_2 \mathbf{A}^2 + a_1 \mathbf{A} + a_o I$, then minimum polynomial is $m(\lambda) = \lambda^3 - a_2 \lambda^2 - a_1 \lambda - a_o$.
- etc.

Trial and Error Method

- By finding factors of $P(\lambda)$, so $m(\lambda)$ is one of the factors of $P(\lambda)$, such that $m(\mathbf{A}) = 0$.

example

- Find minimum polynomial of :

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

Solution:

- The characteristics polynomial of **A** is

$$P(\lambda) = |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 1 & 1-\lambda & 2 \\ 1 & 1 & 2-\lambda \end{vmatrix} = (4 - \lambda)\,\lambda^2$$

- The possibilities of minimum polynomial are : $(4 - \lambda)$, or λ^2, or $(4 - \lambda)\,\lambda$; but only polynomial $m(\lambda) = (4 - \lambda)\,\lambda$ satisfying the condition : $m(\mathbf{A}) = (4\mathbf{I} - \mathbf{A})\,\mathbf{A} = \mathbf{0}$.
- Thus, the minimum polynomial is

$$m(\lambda) = (4 - \lambda)\,\lambda = 4\lambda - \lambda^2$$